Isomorphism between automorphism groups of finitely generated groups

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Abstract. Let G be a finitely generated group and let C^* denote the group of all central automorphisms of G fixing the center of G elementwise. Azhdari and Malayeri [J. Algebra Appl., **6**(2011), 1283-1290] gave necessary and sufficient conditions on G such that $C^* \simeq \text{Inn}(G)$. We prove a technical lemma and, as a consequence, obtain a short and easy proof of this result of Azhdari and Malayeri. Subsequently, we also obtain short proofs of some other existing and some new related results.

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1 Introduction. Let G be a finitely generated group and let $\text{Inn}(G)$ denote the inner automorphism group of G. For normal subgroups X and Y of G, let $\text{Aut}^X(G)$ and $\text{Aut}_Y(G)$ denote the subgroups of $\text{Aut}(G)$ centralizing G/X and Y respectively. We denote the intersection $\text{Aut}^X(G) \cap \text{Aut}_Y(G)$ by $\text{Aut}^X_Y(G)$. Let C^*, in particular, denote the group $\text{Aut}^{Z(G)}_{Z(G)}(G)$, where $Z(G)$ is the center of G. For a finite group G, let G_p and $\pi(G)$ respectively denote the Sylow p-subgroup and the set of prime divisors of G. For a finite p-group G, Attar [2, Main Theorem] proved that $C^* = \text{Inn}(G)$ if and only if either G is abelian or G is nilpotent of class 2 and $Z(G)$ is cyclic. Azhdari and Malayeri [4, Theorem 0.1] (see also [5, Theorem 2.3] for correct version) generalized this result of Attar and proved that if G is a finitely generated nilpotent group of class 2, then $C^* \simeq \text{Inn}(G)$ if and only if $Z(G)$ is infinite cyclic or $Z(G) \simeq C_m \times H \times \mathbb{Z}^r$, where $C_m \simeq \prod_{p \in \pi(G/Z(G))} Z(G)_p$, $H \simeq \prod_{p \notin \pi(G/Z(G))} Z(G)_p$, $r \geq 0$ is the torsion-free rank of $Z(G)$ and $G/Z(G)$ is of finite exponent dividing m. We prove a technical lemma, Lemma 2.1, and as a consequence give a short and easy proof of this main theorem of Azhdari and Malayeri. We also obtain short and alternate proofs of Corollary 2.1 of [5], and Propostion 1.11 and Theorem 2.2(i) of [3]. Some other related results for finitely generated and finite p-groups are also obtained.

By C_p we denote a cyclic group of order p and by X^n we denote the direct product of n-copies of a group X. By $\text{Hom}(G, A)$ we denote the group of all homomorphisms of G into an abelian group A. The rank of G is the smallest cardinality of a generating set of G. The torsion rank and torsion-free rank of G are respectively denoted as $d(G)$ and $\rho(G)$. By $\exp(G)$ we denote the exponent of torsion part of G. All other unexplained

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notations, if any, are standard. The following well known results will be used very frequently without further referring.

Lemma 1.1. *Let U, V and W be abelian groups. Then*
(i) if U is torsion-free of rank m, then $\operatorname{Hom}(U,V) \simeq V^m$, and
(ii) if U is torsion and V is torsion-free, then $\operatorname{Hom}(U,V) = 1$.

2 Main results. Let G be a finitely generated group and M be an abelian subgroup of G with $\pi(M) = \{q_1, q_2, \ldots, q_e\}$. Let L and N be normal subgroups of G such that $G' \leq N \leq L$ and $\pi(G/L) = \pi(G/N) = \{p_1, p_2, \ldots, p_d\}$. Let X, Y, Z be respective torsion parts and a, b, c be respective torsion-free ranks of $G/L, G/N$ and M. Let $X_{p_i} \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$, $Y_{p_i} \simeq \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}$ and $Z_{q_i} \simeq \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$, where for each i, $\alpha_{ij} \geq \alpha_{i(j+1)}$, $\beta_{ij} \geq \beta_{i(j+1)}$ and $\gamma_{ij} \geq \gamma_{i(j+1)}$ are positive integers, respectively denote the Sylow subgroups of X, Y and Z. Then

$$G/L \simeq X \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} X_{p_i} \times \mathbb{Z}^a \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}} \times \mathbb{Z}^a,$$

$$G/N \simeq Y \times \mathbb{Z}^b \simeq \prod_{i=1}^{d} Y_{p_i} \times \mathbb{Z}^b \simeq \prod_{i=1}^{d} \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}} \times \mathbb{Z}^b$$

and

$$M \simeq Z \times \mathbb{Z}^c \simeq \prod_{i=1}^{e} Z_{q_i} \times \mathbb{Z}^c \simeq \prod_{i=1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}} \times \mathbb{Z}^c.$$

Since G/L is a quotient group of G/N, it follows that $a \leq b$, $l_i \leq n_i$ and $\alpha_{ij} \leq \beta_{ij}$ for all $i, 1 \leq i \leq d$ and for all $j, 1 \leq j \leq l_i$. We begin with the following lemma.

Lemma 2.1. *Let G, L, M and N be as above. Then $\operatorname{Hom}(G/N, M) \simeq G/L$ if and only if one of the following conditions hold:*

(i) *G is torsion-free, M is infinite cyclic and both G/L and G/N are torsion-free of same rank.*

(ii) *G is torsion, $M \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i}$, $l_i = n_i$ and either $\alpha_{ij} = \beta_{ij} \leq \gamma_{i1}$ for each j or $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

(iii) *G is a mixed group, $M \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i} \times \mathbb{Z}^c$, both G/L and G/N are finite, $l_i = n_i$ and either $\alpha_{ij} = \beta_{ij} \leq \gamma_{i1}$ for each j or $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

Proof. It is easy to see that if any of the three conditions hold, then $\operatorname{Hom}(G/N, M) \simeq G/L$. Conversely suppose that $\operatorname{Hom}(G/N, M) \simeq G/L$. Then

$$\operatorname{Hom}(Y \times \mathbb{Z}^b, Z \times \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a. \tag{1}$$

We prove only (i) and (ii), because (iii) can be proved using similar arguments. First assume that G is torsion-free. Then N is also torsion-free and therefore by (1) $\operatorname{Hom}(Y \times$

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$\mathbb{Z}^b, \mathbb{Z}^c) \simeq X \times \mathbb{Z}^a$. Thus $X = 1$ and since $a \leq b$, $c = 1$ and $a = b$. It follows that M is infinite cyclic and both G/N and G/L are torsion-free of same rank. Next assume that G is torsion. Then $\text{Hom}(Y, Z) \simeq X$ by (1). Since $\pi(X) = \pi(Y)$ and $d(X) \leq d(Y)$, therefore $q_i = p_i$ and $m_i = 1$ for all $i, 1 \leq i \leq d$. Thus $M \simeq \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}$. Also, observe that

$$
\begin{aligned}
\text{Hom}(Y, Z) &\simeq \text{Hom}(\prod_{i=1}^{d} \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} \prod_{j=1}^{m_i} C_{q_i^{\gamma_{ij}}}) \\
&\simeq \text{Hom}(\prod_{i=1}^{d} \prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, \prod_{i=1}^{d} C_{p_i^{\gamma_{i1}}}) \\
&\simeq \prod_{i=1}^{d} \text{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}})
\end{aligned}
$$

and $X \simeq \prod_{i=1}^{d} \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$. Therefore $\text{Hom}(\prod_{j=1}^{n_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}$ for each $i, 1 \leq i \leq d$, and hence $l_i = n_i$. It thus follows that for each fixed $i, 1 \leq i \leq d$,

$$
\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\alpha_{ij}}}. \tag{2}
$$

Now, if $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$, then $\beta_{ij} \leq \gamma_{i1}$ for each j and $\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}$. It therefore follows from (2) that $\alpha_{ij} = \beta_{ij}$ for each j. And, if $\exp(Y_{p_i}) > \exp(Z_{p_i})$, then there exists largest positive integer r_i between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ and $\beta_{ij} \leq \gamma_{i1}$ for each $j, r_i + 1 \leq j \leq l_i$. Therefore $\text{Hom}(\prod_{j=1}^{l_i} C_{p_i^{\beta_{ij}}}, C_{p_i^{\gamma_{i1}}}) \simeq \prod_{j=1}^{r_i} C_{p_i^{\gamma_{i1}}} \times \prod_{j=r_i+1}^{l_i} C_{p_i^{\beta_{ij}}}$. It then follows by (2) that $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$. $\square$

Remark 2.2. Observe that if $N = L$ and $\exp(G/N) | \exp(M)$, then $\exp(Y_{p_i}) \leq \exp(Z_{p_i})$ for all i and hence $\text{Hom}(G/L, M) \simeq G/L$ if and only if either M is infinite cyclic or $M \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i} \times \mathbb{Z}^c$, where $c \geq 0$ is the torsion-free rank of M.

The next lemma is a little modification of arguments of Alperin [1, Lemma 3] and Fournelle [7, Section 2].

Lemma 2.3. *Let G be any group and Y be a central subgroup of G contained in a normal subgroup X of G. Then the group of all automorphisms of G that induce the identity on both X and G/Y is isomorphic to $\text{Hom}(G/X, Y)$.*

Observe that $C^* \simeq \text{Hom}(G/Z(G), Z(G))$ by Lemma 2.3. If G is nilpotent of class 2, then $\exp(G') = \exp(G/Z(G))$. Now taking $L = M = N = Z(G)$ in Lemma 2.1, we get the following main result of Azhdari and Malayeri [4, Theorem 0.1] (see [5, Theorem 2.3] for correct version).

Corollary 2.4. *Let G be a finitely generated nilpotent group of class 2. Then $C^* \simeq \text{Inn}(G)$ if and only if either $Z(G)$ is infinite cyclic or $Z(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i} \times \mathbb{Z}^c$, where c is the torsion-free rank of $Z(G)$.*

Corollary 2.5 ([5, Corollary 2.1]). *Let G be a finitely generated non-abelian group and let M and N be normal subgroups of G such that $M \leq Z(G) \leq N$ and $G/Z(G)$ is finite. Then $\mathrm{Aut}_N^M(G) = \mathrm{Inn}(G)$ if and only if G is a nilpotent group of class 2, $N = Z(G)$, $G' \leq M$ and $M \simeq C_{\prod_{i=1}^d p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^e Z_{q_i} \times \mathbb{Z}^c$, where $c \geq 0$ is the torsion-free rank of M.*

Proof. First suppose that $\mathrm{Aut}_N^M(G) = \mathrm{Inn}(G)$. Observe that $\mathrm{Aut}_N^M(G) \simeq \mathrm{Hom}(G/N, M)$ by Lemma 2.3. It follows that $\mathrm{Inn}(G)$ is abelian and therefore nilpotence class of G is 2. For any $[a, b] \in G'$, $[a, b] = a^{-1} I_b(a) \in M$ and thus $G' \leq M$. Also, for any $n \in N$, $I_x(n) = n$ for all $x \in G$ and therefore $N = Z(G)$. Now since $\exp(G/Z(G)) = \exp(G')$ divides $\exp(M)$, the result follows from Lemma 2.1 by taking $L = Z(G)$. The converse follows easily. $\square$

In 1911, Burnside [6, Note B, p. 463] gave the notion of pointwise inner automorphism of a group G. An automorphism α of G is called pointwise inner automorphism of G if x and $\alpha(x)$ are conjugate for each $x \in G$. Let H be a characteristic subgroup of G. As defined in [3], an automorphism α of G is called H-pointwise inner if for each element $x \in G$, there exists $h \in H$ such that $\alpha(x) = x^h = x[x, h]$. For convenience, we denote $\gamma_k(G)$-pointwise inner automorphism of G by $\mathrm{Aut}_{k-pwi}(G)$. As another application of Lemma 2.1, we get the following two results of Azhdari [3]. The second one generalizes Theorem 2.2(i) of [3].

Corollary 2.6 ([3, Prop. 1.11]). *Let G be a finitely generated nilpotent group of class $k + 1 \geq 2$. Then $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq G/\zeta_k(G)$ if and only if $\gamma_{k+1}(G)$ is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then $\mathrm{Aut}_{k-pwi}(G)$ is isomorphic to a quotient group of $\mathrm{Inn}(G)$.*

Proof. It follows from [9, Cor. 2.6, Cor. 3.16, Cor. 3.17] that $\exp(G/\zeta_k(G)) = \exp(\gamma_{k+1}(G))$ and $G/\zeta_k(G)$ is finite if and only if $\gamma_{k+1}(G)$ finite. The result now follows from Lemma 2.1 (see Remark 2.2) by taking $L = N = \zeta_k(G)$ and $M = \gamma_{k+1}(G)$. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$ is cyclic, then using the arguments as in [10, Prop. 3.1], we can prove that $\mathrm{Aut}_{k-pwi}(G) \simeq \mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G))$. $\square$

Corollary 2.7 (*cf.* [3, Theorem 2.2(i)]). *Let G be a finitely generated nilpotent group of class $k + 1 \geq 2$. Then $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \mathrm{Inn}(G)$ if and only if G is nilpotent of class 2 and G' is cyclic. In particular, if $\gamma_{k+1}(G) = [x, \gamma_k(G)]$ for all $x \in G \setminus C_G(\gamma_k(G))$, then $\mathrm{Aut}_{k-pwi}(G) \simeq \mathrm{Inn}(G)$ if and only if G is nilpotent of class 2 and G' is cyclic.*

Proof. Observe that if $\mathrm{Hom}(G/\zeta_k(G), \gamma_{k+1}(G)) \simeq \mathrm{Inn}(G)$, then $G/Z(G)$ is abelian, and therefore nilpotence class of G is 2. It follows that $\zeta_k(G) = Z(G)$ and $\gamma_{k+1}(G) = G'$. The result now follows from above corollary by taking $k = 1$. $\square$

For $g \in G$ and $\alpha \in \mathrm{Aut}(G)$, the element $[g, \alpha] = g^{-1}\alpha(g)$ is called the autocommutator of g and α. Inductively, define

$$[g, \alpha_1, \alpha_2, \ldots, \alpha_n] = [[g, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}], \alpha_n],$$

where $\alpha_i \in \mathrm{Aut}(G)$. The absolute center $L(G)$ of G is defined as

$$L(G) = \{g \in G \mid [g, \alpha] = 1, \text{ for all } \alpha \in \mathrm{Aut}(G)\}.$$

Let $L_1(G) = L(G)$, and for $n \geq 2$, define $L_n(G)$ inductively as

$$L_n(G) = \{g \in G \mid [g, \alpha_1, \alpha_2, \ldots, \alpha_n] = 1 \text{ for all } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathrm{Aut}(G)\}.$$

The autocommutator subgroup G^* of G is defined as

$$G^* = \langle g^{-1}\alpha(g) \,|\, g \in G, \alpha \in \text{Aut}(G)\rangle.$$

It is easy to see that $L_n(G) \leq Z_n(G)$ for all $n \geq 1$ and $G' \leq G^*$. An automorphism α of G is called an autocentral automorphism if $g^{-1}\alpha(g) \in L(G)$ for all $g \in G$. The group of all autocentral automorphisms of G is denoted by $\text{Var}(G)$. A group G is called autonilpotent of class at most n if $L_n(G) = G$ for some natural number n. Observe that if G is autonilpotent of class 2, then $G^* \leq L(G)$. Nasrabadi and Farimani [8] proved that if G is a finie autonilpotent p-group of class 2, then $\text{Var}(G) = \text{Inn}(G)$ if and only if $L(G) = Z(G)$ and $Z(G)$ is cyclic. Observe that $\text{Var}(G) \simeq \text{Hom}(G/L(G), L(G))$ by Lemma 2.3. As a final consequence of Lemma 2.1, we get the following result which generalizes the main result of Nasrabadi and Farimani. The proof follows from Lemma 2.1 by taking $M = N = L(G)$ and $L = Z(G)$.

Corollary 2.8. *Let G be a finitely generated non-abelian group such that $G' \leq L(G)$ and $\pi(G/L(G)) = \pi(G/Z(G))$. Then $\text{Var}(G) \simeq \text{Inn}(G)$ if and only if one of the following holds*

(i) *G is torsion-free, $L(G)$ is infinite cyclic and $\rho(G/L(G)) = \rho(G/Z(G))$;*

(ii) *G is torsion, $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i}$ and either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

(iii) *G is a mixed group, both $G/L(G)$ and $G/Z(G)$ are finite, $L(G) \simeq C_{\prod_{i=1}^{d} p_i^{\gamma_{i1}}} \times \prod_{i=d+1}^{e} Z_{q_i} \times \mathbb{Z}^c$ and either $L(G) = Z(G)$ or $l_i = n_i$, $\alpha_{ij} = \gamma_{i1}$ for $1 \leq j \leq r_i$ and $\alpha_{ij} = \beta_{ij}$ for $r_i + 1 \leq j \leq l_i$, where r_i is the largest positive integer between 1 and l_i such that $\beta_{ir_i} > \gamma_{i1}$ for each fixed $i, 1 \leq i \leq d$.*

Let G be a finite p-group such that $G' \leq L(G)$. Let $G/Z(G) \simeq \prod_{i=1}^{r} C_{p^{\alpha_i}}$, $G/L(G) \simeq \prod_{i=1}^{s} C_{p^{\beta_j}}$ and $L(G) \simeq \prod_{i=1}^{t} C_{p^{\gamma_i}}$, where $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_r$, $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_s$ and $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_t$ are positive integers. Since $G/Z(G)$ is a quotient group of $G/L(G)$, $r \leq s$ and $\alpha_i \leq \beta_i$ for $1 \leq i \leq r$.

Corollary 2.9. *Let G be a finite non-abelian p-group. Then $\text{Var}(G) = \text{Inn}(G)$ if and only if $G' \leq L(G)$, $L(G)$ is cyclic and either $L(G) = Z(G)$ or $d(G/L(G)) = d(G/Z(G))$, $\alpha_i = \gamma_1$ for $1 \leq i \leq k$ and $\alpha_i = \beta_i$ for $k + 1 \leq i \leq r$, where k is the largest positive integer such that $\beta_k > \gamma_1$.*

Proof. Observe that if $\text{Var}(G) = \text{Inn}(G)$, then for any $[a, b] \in G'$, $[a, b] = a^{-1}I_b(a) \in L(G)$ and thus $G' \leq L(G)$. The result now follows from Cor. 2.8. □

Corollary 2.10 ([8, Theorem 3.2]). *Let G be a non-abelian autonilpotent finite p-group of class 2. Then $\text{Var}(G) = \text{Inn}(G)$ if and only if $L(G) = Z(G)$ and $L(G)$ is cyclic.*

Proof. Suppose that $\text{Var}(G) = \text{Inn}(G)$. Observe that if $g^{-1}\alpha(g) \in G^*$, then $\alpha(g) = gl$ for some $l \in L(G)$ and hence $(g^{-1}\alpha(g))^m = g^{-m}\alpha(g)^m$ for all $m \geq 1$. Let $\exp(G/L(G)) = d$ and $\exp(G^*) = k$. Then $1 = (g^{-1}\alpha(g))^k = g^{-k}\alpha(g)^k$ implies that $g^k \in L(G)$ and hence $d \leq k$. Conversely, if $gL(G) \in G/L(G)$, then $g^d \in L(G)$ and thus $1 = g^{-d}\alpha(g^d) = (g^{-1}\alpha(g))^d$. It follows that $k \leq d$ and hence $\exp(G/L(G)) = \exp(G^*)$. Since $G^* \leq L(G)$, $\exp(G/L(G))| \exp(L(G))$. Therefore $\text{Var}(G) \simeq \text{Hom}(G/L(G), L(G)) \simeq G/L(G)$, because $L(G)$ is cyclic by Corollary 2.9, and hence $L(G) = Z(G)$. □

References

[1] J. L. Alperin, *Groups with finitely many automorphisms*, Pacific J. Math., **12**(1962), 1-5.

[2] M. S. Attar, *On central automorphisms that fix the centre elementwise*, Arch. Math., **89**(2007), 296-297.

[3] Z. Azhdari, *On certain automorphisms of nilpotent groups*, Math. Proc. R. Ir. Acad., **113**A(2013), 5-17.

[4] Z. Azhdari, M. A. Malayeri, *On inner automorphisms of nilpotent group of class* 2, J. Algebra Appl., **6**(2011), 1283-1290.

[5] Z. Azhdari, M. A. Malayeri, *On automorphisms fixing certain groups*, J. Algebra Appl., **2**(2013), 1250163(1-17).

[6] W. Burnside, *Theory of Groups of Finite Order*, 2nd ed., Cambridge University Press, Cambridge, 1911.

[7] T. A. Fournelle, *Torsion in semicomplete nilpotent groups*, Math. Proc. Cambridge Philos. Soc., **94** (1983), 191-202.

[8] M. M. Nasrabadi, Z. K. Farimani, *Absolute central automorhisms that are inner*, Indag. Math. (N.S.), **26**(2015), 137-141.

[9] R. B. Warfield Jr., *Nilpotent Groups*, Lecture Notes in Mathematics, vol. 513, Springer-Verlag, New York, 1976.

[10] M. K. Yadav, *On automorphisms of some finite p-groups*, Proc. Indian Acad. Sci. (Math. Sci.), **118**(1)(2008), 1-11.